Plasma electric microfield distribution with the triplet correlation contribution: High-frequency component at a neutral point

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(Received 6 May 2004; published 8 October 2004)

We report on nontrivial calculations of a triplet correlation contribution to the high-frequency component of the microfield distribution at a neutral particle point by utilizing an expansion, initially proposed by Baranger and Mozer, and demonstrate that its contribution leads to a shift of the maximum in the distribution to weaker fields relative to results including the pair correlation only. The entire picture of the high-frequency microfield distribution is studied and detailed comparison with the results of Hooper is made.

DOI: 10.1103/PhysRevE.70.046404 PACS number(s): 52.25.−b, 52.27.Aj, 52.27.Gr

I. INTRODUCTION

In real plasmas various elementary processes, such as, for instance, an excitation of electrons in atoms and ions of the matter, permanently occur. The excited atoms and ions then get rid of the surplus energy by radiating electromagnetic waves (photons). It is natural that all processes, being in a plasma, may, at a different extent, influence its characteristics. On the other hand, all the phenomena, taking place in the plasma medium, are affected by its electric microfield, i.e., the electric field produced by all charges in the plasma, which, thus, is of great importance in the theory and applications. In particular, when applied to the process of the radiation of excited atoms and ions, the electric microfield is known to be responsible for the so-called Stark broadening of spectral lines.

The problem of determining electric microfield distributions is conventionally divided into two parts due to the existence of two different time scales in plasmas. On the time scale comparable to the electron relaxation time the plasma medium may be considered a gas of electrons immersed in a positively charged neutralizing background of ions and, thus, the Coulomb forces act at the observation point to generate the microfield distribution which is then called the high-frequency component since the electron relaxation time is dramatically less in magnitude than the ion one. The low-frequency component of the microfield, appearing on the ion relaxation time scale, is then introduced by the notion that it is governed by the dynamics of ions, surrounded by electron clouds, and, consequently, the shielded Coulomb forces at the observation point should thoroughly be considered. Since the theory of the microfield distribution is chiefly supposed to be applicable to the problem of spectral line broadening an important question is what is it at the observation point. A particle at the observation point is called a radiator and if it is an ion, its correlation with the surrounding plasma medium should necessarily be taken into account, whereas if it is an atom, no correlation ties it to neighboring plasma particles.

Since the work of Holtsmark [1], who completely neglected correlations between particles, most of all efforts have been concentrated on a theory of the microfield distribution with inclusion of collective events in plasmas. The first remarkable advance in this direction was made by Baranger and Mozer [2,3], who wrote the distributions of both high- and low-frequency components of the microfield distribution as expansions with respect to the correlation functions which then had been terminated at the pair correlation. To do that they applied the Debye-Hückel form of the pair correlation function that corresponds to the first order of the expansion in a nonideality parameter. It was argued however that such an approach is only valid for low-density, high-temperature plasmas where a deviation from the Holtsmark original distribution, corresponding to the first term in the series, is not large. Afterwards Hooper and Tighe [4–6] reformulated this expansion in terms of other functions by introducing a free parameter which had been chosen on the basis of an argument of arriving at a plateau where there was no dependence on the free parameter itself. To improve these results Iglesias and Hooper [7] included in the analysis the Debye-chain cluster expansion similar to that of Ursell and Mayer [8]. Quite an analogous approach, now known as APEX, was proposed by Iglesias, Lebowitz *et al.* [9–11] but the free parameter, called adjustable, had been picked out to satisfy the exactly known second moment rule for the electric field strength. In the early 1980s, following the idea of Morita on the similarity of the representation of the microfield distribution to that of the excess chemical potential, Iglesias [12] virtually reduced the problem to determination of the radial distribution function for a fictious system with an imaginary part of the interaction potential energy. Employing this idea allowed Lado [13,14] to develop an integral equation technique for calculating the radial distribution function and good agreement was discovered with computer simulations.

Another point of interest is an inclusion of quantum mechanical effects. It was done by Held and co-workers [15–17] at high temperatures when the Landau length is

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smaller than the thermal de Broglie wavelength of electrons. In this particular low-density case it becomes possible to utilize a semiclassical approach with a properly defined pairwise pseudopotential model [18–20]. The pseudopotentials and the corresponding correlation functions were then used in the framework of the Baranger-Mozer expansion to find both a very rich picture for the microfield distribution behavior, depending on plasma parameters chosen, and agreement with another approach as well [21]. One of the other possible approaches to the problem is the density-response formalism of quantum liquids. The first attempt in this direction was undertaken by Iglesias and Hooper [22], who assumed the response function to acquire a form of the random phase approximation. Later Ichimaru and Yan [23] extended this approach to strongly coupled plasmas by introducing a local field correction. Another very sophisticated method was put forward by Perrot and Dharma-Wardana [24]. It is based on the density functional theory with partial resummation of higher-order terms in the Baranger-Mozer series [25] and takes into consideration such factors as an internal structure of the test particle and strong coupling. More detailed analysis of these and other aspects of the microfield distribution problem may be found in the exhaustive review [26].

When the motion of perturbing ions and the radiator cannot be ignored any longer during the radiation process of the test particle an appropriate consideration of dynamical effects turns essential [27,28]. On the other hand effects of strong coupling can establish nonuniformity of the electric microfield distribution caused by multipole moments of the plasma system [29–33]. That is why, at present, there is a trend to incorporate dynamical effects and strong coupling phenomena as well which can give rise to the nonuniformity of the electric microfield distribution and, as a result, to the asymmetry of spectral line shapes.

It is worthwhile emphasizing that until now most of all works accumulated the pair correlation approach when two perturbers create the microfield distribution at the observation point and are correlated with each other and with the radiator due to the reciprocal interaction of plasma particles. In contrast, we report on direct calculations in which a third perturber is involved. For this purpose we choose the technique, proposed by Baranger and Mozer, because it is essentially free of any crucial physical assumption and, thus, it is mostly as primary as computer simulations. The only restriction comes from the fact that the Baranger-Mozer expansion is implicitly a series in the correlation parameter introduced beneath and, therefore, the range of its validity is bounded on the weakly coupled regime.

The sketch of this communication is outlined as follows. In the following section, dimensionless parameters and values, suitable for the problem of the microfield distribution, are introduced. Section III describes the basic formalism needed for all further consideration. Main results and discussions are stated in Sec. IV to demonstrate the significance of handling higher order terms in the Baranger-Mozer series. Section V concludes this paper by main inferences and provisions for future work as well.

II. DIMENSIONLESS PARAMETERS AND MAGNITUDES

Of interest hereinafter is the high-frequency component of the microfield distribution and, as is aforesaid, this means that the system under consideration is the one-component, electron plasma, i.e., a gas of electrons moving in a uniform neutralizing background of positively charged ions.

The first value relevant to the problem of the microfield distribution is the distance a_0 very close, in sense of magnitude, to the mean interelectron spacing and it is defined as

$$\tfrac{4}{15}(2\pi)^{3/2} n a_0^3 = 1, \tag{1}$$

where n denotes the equilibrium electron number density. Using Eq. (1) the electric field strength is suitably normalized through

$$\beta = \frac{E}{E_0}. \tag{2}$$

Here $E_0 = e/a_0^2$ and e refers to the magnitude of the elementary electric charge.

Despite the definition in Eq. (1) all distances and vectors in what follows are made dimensionless by using the Debye electron length $\lambda_D = (k_B T/4\pi n e^2)^{1/2}$ as

$$\mathbf{x} = \frac{\mathbf{r}}{\lambda_D} \quad \text{or } x = \frac{r}{\lambda_D}, \tag{3}$$

where k_B designates the Boltzmann constant and T signifies the electron temperature.

To measure importance of interparticle correlations in plasmas the correlation parameter y is merely introduced via the ratio of the parameter a_0 and the Debye electron length λ_D,

$$y = \frac{a_0}{\lambda_D}, \tag{4}$$

and it is easily related to the standard nonideality plasma parameter, defined through the ratio of the Landau length $a_L = e^2/k_B T$ and the Debye screening radius λ_D or to the more usual Coulomb coupling parameter $\Gamma = a_L(4\pi n/3)^{1/3}$ as

$$\Lambda = \frac{a_L}{\lambda_D} = \frac{\sqrt{8\pi}}{15} y^3 \quad \text{and } \Gamma = a_L\left(\frac{4\pi n}{3}\right)^{1/3} = \left(\frac{8\pi}{25}\right)^{1/3} \frac{y^2}{3}. \tag{5}$$

From Eq. (5) one can conclude that the inequality $y \leqslant 1.5$ (i.e., $\Lambda \leqslant 1.128$ or $\Gamma \leqslant 0.751$) stands for weakly or even moderately coupled regimes which are of particular interest herein.

III. BASIC FORMALISM

In the most general case of physical interest the microfield distribution $W(\mathbf{E})$ is written via the probability density $P_{N+1}(\mathbf{r_0}, \mathbf{r_1}, \ldots, \mathbf{r_N})$ of finding a certain configuration $\mathbf{r_0}, \mathbf{r_1}, \ldots, \mathbf{r_N}$ of $N+1$ particles as

$$W(\mathbf{E}) = \int \cdots \int \delta\left(\mathbf{E} - \sum_{i=1}^{N} \mathbf{E_i}\right) P_{N+1}(\mathbf{r_0}, \mathbf{r_1}, \cdots, \mathbf{r_N}) \times d\mathbf{r_0}\, d\mathbf{r_1} \cdots d\mathbf{r_N}, \tag{6}$$

where $\mathbf{E_i}$ is the electric field exerted by the ith particle at the site of the radiator located at $\mathbf{r_0}$.

For a neutral particle at the observation point $\mathbf{r_0}$ there is no correlation between the radiator and other plasma particles and, thus, using the obvious relation $P_{N+1}(\mathbf{r_0}, \mathbf{r_1}, \ldots, \mathbf{r_N}) = P_1(\mathbf{r_0}) P_N(\mathbf{r_1}, \ldots, \mathbf{r_N})$ with the single particle distribution function $P_1(\mathbf{r_0})$, Eq. (6), after placing the origin of the coordinates at the radiator site, simplifies to

$$W(\mathbf{E}) = \int \cdots \int \delta\left(\mathbf{E} - \sum_{i=1}^{N} \mathbf{E_i}\right) \times P_N(\mathbf{r}_1, \ldots, \mathbf{r_N}) d\mathbf{r}_1 \cdots d\mathbf{r_N}, \tag{7}$$

where the Coulombic form of the electric field strength

$$\mathbf{E_i} = \frac{e}{r_i^3} \mathbf{r_i} \tag{8}$$

is conventionally conceded to evaluate the high-frequency component of the electric microfield in classical plasmas.

It turns out convenient to operate with the Fourier transform $F(\mathbf{k})$ of $W(\mathbf{E})$ and, as was shown by Baranger and Mozer [2], an expansion in the correlation functions $g_p(\mathbf{r_1}, \ldots, \mathbf{r_p})$ yields the following exponential series for $F(\mathbf{k})$:

$$F(\mathbf{k}) = \exp\left[\sum_{p=1}^{\infty} \frac{n^p}{p!} h_p(\mathbf{k})\right], \tag{9}$$

where $h_p(\mathbf{k})$ is expressed through the correlation function $g_p(\mathbf{r_1}, \ldots, \mathbf{r_p})$ of p particles as

$$h_p(\mathbf{k}) = \int \cdots \int \varphi_1 \ldots \varphi_p g_p(\mathbf{r}_1, \ldots, \mathbf{r_p}) d\mathbf{r}_1 \cdots d\mathbf{r_p} \tag{10}$$

with the definition

$$\varphi_j = \exp(i\mathbf{k} \cdot \mathbf{E_j}) - 1. \tag{11}$$

Due to its isotropy the function $F(\mathbf{k})$ depends on the module of the vector $\mathbf{k}$ only. Introducing the dimensionless vector $\mathbf{u} = \mathbf{k} E_0$ and truncating series (9) at the third term produce

$$F(u) = \exp\left[n h_1(u) + \frac{n^2}{2!} h_2(u) + \frac{n^3}{3!} h_3(u)\right] \tag{12}$$

with the Holtsmark contribution [1]

$$n h_1(u) = -u^{3/2}, \tag{13}$$

the Baranger and Mozer contribution [2,3]

$$\frac{n^2}{2!} h_2(u) = \frac{1}{2!(4\pi\Lambda)^2} \int \int \varphi_1 \varphi_2 g_2(\mathbf{x_1}, \mathbf{x_2}) d\mathbf{x_1}\, d\mathbf{x_2}, \tag{14}$$

and the triplet correlation contribution

$$\frac{n^3}{3!} h_3(u) = \frac{1}{3!(4\pi\Lambda)^3} \int \int \int \varphi_1 \varphi_2 \varphi_3 \times g_3(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) d\mathbf{x_1}\, d\mathbf{x_2}\, d\mathbf{x_3}, \tag{15}$$

with

$$\varphi_j = \exp\left(i y^2 \frac{\mathbf{u} \cdot \mathbf{x_j}}{x_j^3}\right) - 1. \tag{16}$$

Equation (12) is valid for a weakly coupled plasma and a number of terms to be taken into account in series (9) grows when the correlation parameter increases. Following the same idea, the functions $g_p(\mathbf{r_1}, \ldots, \mathbf{r_p})$ can also be expanded with respect to the nonideality parameter Λ and a resulting series can then be truncated at the desirable order and, of course, such a method for evaluating correlation functions is quite consistent with Eq. (12). To do all that just mentioned we use the cluster expansion for the pair correlation function at order Λ^2 [34–38],

$$g_2(\mathbf{x_1}, \mathbf{x_2}) = -\Lambda g_{2,\Lambda}(\mathbf{x_1}, \mathbf{x_2}) + \Lambda^2 g_{2,\Lambda^2}(\mathbf{x_1}, \mathbf{x_2}) = -\Lambda \Phi(x_{12}) + \frac{\Lambda^2}{2!} \Phi(x_{12})^2 - \frac{\Lambda^2}{2x_{12}} [\ln 3 \exp(-x_{12}) + \exp(-x_{12}) \mathrm{Ei}(-x_{12}) - \exp(x_{12}) \mathrm{Ei}(-3x_{12})] + \frac{\Lambda^2}{8x_{12}} \left[\ln 3 (1 + x_{12}) \exp(-x_{12}) - \frac{4}{3} \{\exp(-x_{12}) - \exp(-2x_{12})\} + (1 + x_{12}) \exp(-x_{12}) \mathrm{Ei}(-x_{12}) - (1 - x_{12}) \exp(x_{12}) \mathrm{Ei}(-3x_{12})\right] \tag{17}$$

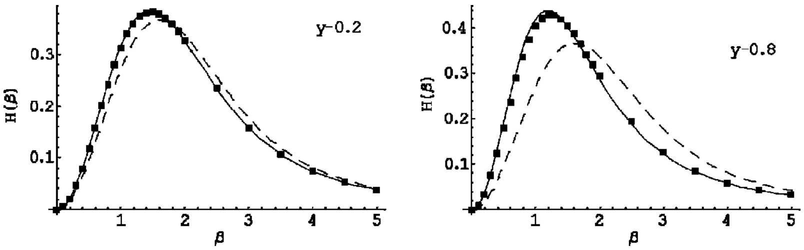

FIG. 1. The high-frequency microfield distribution in Baranger and Mozer series (9) terminated at the third term. Dashed line, Holtsmark [1]; black squares, Hooper [4]; solid line, the third term truncation of the Baranger and Mozer series (present results).

together with the convolution approximation for the triplet correlation function at the same order [39,40]

$$g_3(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) = \Lambda^2\Phi(x_{12})\Phi(x_{13}) + \Lambda^2\Phi(x_{12})\Phi(x_{23}) + \Lambda^2\Phi(x_{13})\Phi(x_{23}) - \frac{\Lambda^2}{4\pi}\int \Phi(x_{14})\Phi(x_{24})\Phi(x_{34})d\mathbf{x_4}. \quad (18)$$

Here we introduced the notation $x_{ij}=|\mathbf{x_i}-\mathbf{x_j}|$, the Debye-Hückel function $\Phi(x_{ij})=\exp(-x_{ij})/x_{ij}$ and the exponential integral of the form

$$\mathrm{Ei}(-x) = -\int_x^\infty \frac{\exp(-t)}{t}dt. \quad (19)$$

Simplifying the expressions (17) and (18) go absolutely in line with the truncation of series (9) since the entire technique developed is aimed at the weakly coupled regime where the Debye type theory (17) and the convolution approximation (18) should certainly be valid and the first neglected term $g_4(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3},\mathbf{x_4})$ is of the higher order Λ^3.

Even employing Eqs. (17) and (18) does not make the integrals in Eqs. (14) and (15) able to be evaluated and further simplification is achieved by expanding integrands in the spherical harmonics $Y_{lm}(\theta,\omega)$. The details and subtleties of this routine procedure are transferred to the Appendix where it is shown how to handle new contributions beyond Baranger and Mozer [2].

Taking into account the isotropy of the microfield distribution its original form is then restored from the reversed Fourier transformation as

$$H(\beta) = 4\pi\beta^2 W(\beta) = \frac{2\beta}{\pi}\int_0^\infty u\sin(\beta u)F(u)du. \quad (20)$$

IV. RESULTS AND DISCUSSIONS

Numerical calculations have been made for the high frequency microfield distribution equations (12) and (20). Series expansions in spherical harmonics converge very rapidly and were all terminated at $l=2$. As was first noted in Ref. [41] there is a numerical error in the calculations of Baranger and Mozer [2] and we can confirm now that the computations in Ref. [41] are quite accurate. It is therefore sensible to make a comparison of the present results with the data available, for example, in Ref. [4]. To do so Fig. 1 is plotted for two values of the correlation parameter y. From this figure one can infer that the deviation of Hooper's results from those in the Baranger and Mozer formalism is indeed very small relative to their shift from the Holtsmark distribution as both use cluster-type expansions although determined in different functions but quite consistent in spirit.

One can see that there is a maximum in the curve of the microfield distribution of Holtsmark. The numerical analysis implemented herein makes it clear that taking into account higher order correlation effects will not alter the qualitative picture as a whole and, thus, there must be a maximum in the curve of the microfield distribution in the most general situation. Figure 1 and results of many other authors as well [4–17,21–25] definitely show that it is the case. At the same time one can observe that correlation effects shift the maximum of the distribution to weaker fields relative to the Holtsmark distribution which is also clear because those correlation effects in the electron-electron repulsive interaction make the probability density respect larger distances between the particles and, consequently, weaker fields. The generality of such an argument demonstrates unambiguously that the correlation phenomena should always shift the microfield distribution to weaker fields in comparison with the Holtsmark theory.

In Fig. 2 the detailed comparison is made of the second and third term truncation of the Baranger and Mozer series.

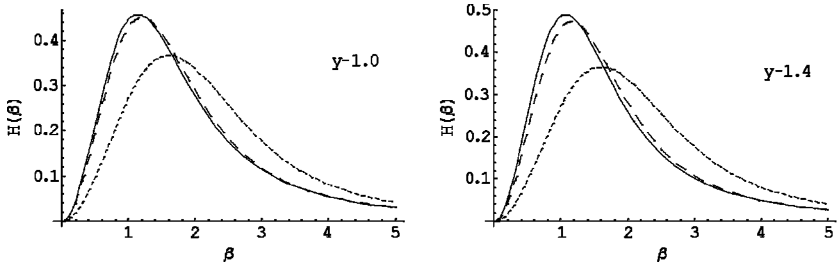

FIG. 2. The high-frequency microfield distribution in Baranger and Mozer series (9) terminated at the third term. Dashed line, the second term truncation of the Baranger and Mozer series (9); dotted line, Holtsmark [1]; solid line, the third term truncation of the Baranger and Mozer series (present results).

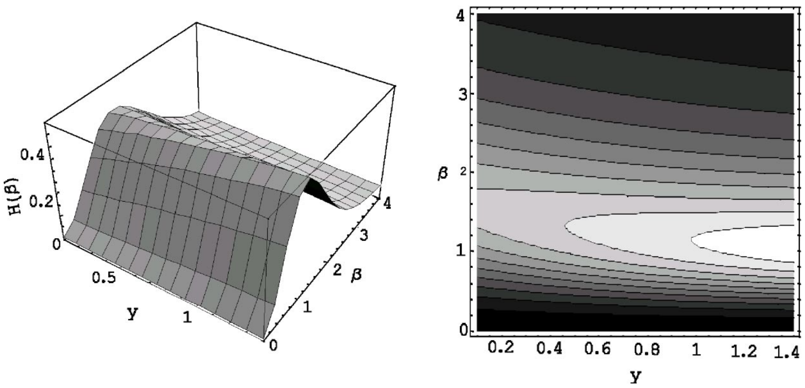

FIG. 3. The entire picture of the high-frequency microfield distribution $H(\beta,y)$ in Baranger and Mozer series (9) terminated at the third term.

For this particular domain it is seen that the discrepancy is not very large and the difference between the approaches is pure quantitative but it becomes more noticeable for larger values of the correlation parameter. As also evidenced by computer simulations one can envisage that higher order correlations lead to a shift of the maximum in the distribution to weaker fields, which grows while the correlation parameter increases.

To observe the entire picture of the high-frequency microfield distribution for a range of the correlation parameter $y=0-1.4$ Fig. 3 is shown. It includes the 3D graph of the distribution against both the electric field strength β and the correlation parameter y, and the corresponding contour plot, showing in the y-β plane the curves of the same height in the 3D surface, to identify the location of the maximum. The more versatile behavior of the microfield distribution is observed in case of the third term truncation of the Baranger and Mozer series with the shift of the maximum to weaker field strengths.

V. CONCLUSIONS

This work has considered the third term truncation (due to the triplet correlations) of the Baranger and Mozer series for the high-frequency microfield distribution at a neutral particle point. It has been shown that such an approach agrees well with the results of Hooper as both use cluster-type expansions to treat the problem in hand. For large enough values of the correlation parameter it has been found possible to observe larger deviations of the third term truncation from the second one with the shift to smaller values of the electric field strength.

To be strict the method described above is only valid for the weak coupling regime but fails to predict correct values for the strongly coupled plasmas where computer simulations (Monte Carlo and molecular dynamics) should certainly work well. On the other hand, it is absolutely deprived of any essential physical assumption and is, thus, as fundamental as computer simulations based on first principles. Consequently, the developed technique together with the computer simulation methods, for which the limit to the weak coupling regime is difficult to trace down, lays the firm foundation for studying the microfield distribution picture in a wide range of plasma parameters.

It should also be noted here that in all the cases considered above the asymptotics at $\beta\to\infty$ makes the second moment of the electric field strength, which is important from the viewpoint of the energy preserved in the microfield, diverge for a neutral particle point. Quantum mechanical statistics will not seem to facilitate this situation since, as was shown in Ref. [42], that divergence is caused by the unphysical assumption on the pointlike electric charges.

There are several ways to extend the technique used herein to larger values of the coupling parameter and to expand it to other situations of physical interest. First of all, to avoid the resummation like in Eq. (17) it is desirable to get the hypernetted chain approximation involved into the approach. The other interesting objective is to consider the low-frequency component of the microfield distribution and to embody quantum mechanical effects via the pseudopotential model developed, for instance, in Refs. [18–20].

ACKNOWLEDGMENTS

One of the authors (A.D.) gratefully acknowledges the financial support from the NATO grant service system and thanks the Laboratoire de Physique des Gaz et des Plasmas of the Université de Paris-Sud for the hospitality rendered him during his stay there.

APPENDIX

1. Second term truncation of the Baranger and Mozer series

All the functions appearing in integrals (14) and (15) are expanded in spherical harmonics as follows:

$$\varphi_j = \sum_{l=0}^{\infty} i^l [4\pi(2l+1)]^{1/2} \left[J_l\left(\frac{y^2 u}{x_j^2}\right) - \delta_{l0} \right] Y_{l0}(\theta_j,\omega_j), \tag{A1}$$

$$\begin{aligned} g_{2,\Lambda}(\mathbf{x_i},\mathbf{x_j}) &= \Phi(x_{ij}) \\ &= \sum_{l=0}^{\infty} \left[\frac{2l+1}{4\pi}\right]^{1/2} f_l(x_i,x_j) Y_{l0}(\theta_{ij},\omega_{ij}) \\ &= \sum_{l,m} f_l(x_i,x_j) Y^*_{lm}(\theta_i,\omega_i) Y_{lm}(\theta_j,\omega_j), \end{aligned} \tag{A2}$$

$$g_{2,\Lambda^2}(\mathbf{x_i},\mathbf{x_j}) = \sum_{l=0}^{\infty}\left[\frac{2l+1}{4\pi}\right]^{1/2} r_l(x_i,x_j)Y_{l0}(\theta_{ij},\omega_{ij}) = \sum_{l,m} r_l(x_i,x_j)Y^*_{lm}(\theta_i,\omega_j)Y_{lm}(\theta_i,\omega_j), \quad \text{(A3)}$$

where δ refers to the Kronecker delta, J_l designates the Bessel spherical function and the expressions for $f_l(x_i,x_j)$ and $r_l(x_i,x_j)$ are written as

$$f_l(x_i,x_j) = 2\pi\int_{-1}^{+1} d\mu\, P_l(\mu)\Phi(x_{ij}), \quad \mu=\cos(\mathbf{x_i}\cdot\mathbf{x_j}), \quad \text{(A4)}$$

$$r_l(x_i,x_j) = 2\pi\int_{-1}^{+1} d\mu\, P_l(\mu)g_{2,\Lambda^2}(\mathbf{x_i},\mathbf{x_j}), \quad \mu=\cos(\mathbf{x_i}\cdot\mathbf{x_j}), \quad \text{(A5)}$$

with the Legendre polynomial P_l.

The functions $f_l(x_i,x_j)$ and $r_l(x_i,x_j)$ may neatly be derived in an analytical form and for $l=0$, 1, 2 are concisely written as follows:

$$f_0(x_i,x_j) = \frac{2\pi}{x_ix_j}[\exp(-|x_i-x_j|)-\exp(-x_i-x_j)], \quad \text{(A6)}$$

$$f_1(x_i,x_j) = \frac{2\pi}{x_i^2x_j^2}[(x_ix_j-|x_i-x_j|-1)\exp(-|x_i-x_j|) + (x_ix_j+x_i+x_j+1)\exp(-x_i-x_j)], \quad \text{(A7)}$$

$$f_2(x_i,x_j) = \frac{2\pi}{x_i^2x_j^2}\left[\left(x_ix_j-3-3|x_i-x_j|+\frac{3(x_i-x_j)^2}{x_ix_j}+\frac{9|x_i-x_j|}{x_ix_j}+\frac{9}{x_ix_j}\right)\exp(-|x_i-x_j|) - \left(x_ix_j+3+3(x_i+x_j)+\frac{3(x_i+x_j)^2}{x_ix_j} + \frac{9(x_i+x_j)}{x_ix_j}+\frac{9}{x_ix_j}\right)\exp(-x_i-x_j)\right], \quad \text{(A8)}$$

$$r_0(x_i,x_j) = \frac{\pi}{12x_ix_j}[2\exp(-2|x_i-x_j|) - 2\exp(-2(x_i+x_j)) - \exp(-|x_i-x_j|)(4+3\ln 3[2-|x_i-x_j|]) + \exp(-(x_i+x_j))(4+3\ln 3[2-(x_i+x_j)]) - 3\exp(-|x_i-x_j|)\mathrm{Ei}(-|x_i-x_j|)(2-|x_i-x_j|) + 3\exp(-(x_i+x_j)) \times\mathrm{Ei}(-(x_i+x_j))(2-(x_i+x_j)) - 3\exp(|x_i-x_j|)\mathrm{Ei}(-3|x_i-x_j|)(2+|x_i-x_j|) + 3\exp(x_i+x_j)\mathrm{Ei}(-3(x_i+x_j)) \times(2+(x_i+x_j))], \quad \text{(A9)}$$

$$r_1(x_i,x_j) = \frac{\pi}{12x_i^2x_j^2}[2\exp(-2(x_i+x_j))(1+2(x_i+x_j)+x_ix_j) - 2\exp(-2|x_i-x_j|)(1+2|x_i-x_j|-x_ix_j) - \exp(-(x_i+x_j))(4[1+(x_i+x_j)+x_ix_j] - 3\ln 3[x_i^2+x_j^2+x_ix_j(x_i+x_j)]) + \exp(-|x_i-x_j|)(4[1+|x_i-x_j|-x_ix_j] - 3\ln 3[x_i^2+x_j^2-x_ix_j|x_i-x_j|]) + 3\exp(-(x_i+x_j))\mathrm{Ei}(-(x_i+x_j))(x_i^2+x_j^2+x_ix_j(x_i+x_j)) - 3\exp(-|x_i-x_j|)\mathrm{Ei}(-|x_i-x_j|)(x_i^2+x_j^2-x_ix_j|x_i-x_j|) + 3\exp(x_i+x_j)\mathrm{Ei}(-3(x_i+x_j))(x_i^2+x_j^2-x_ix_j(x_i+x_j)) - 3\exp(|x_i-x_j|)\mathrm{Ei}(-3|x_i-x_j|)(x_i^2+x_j^2+x_ix_j|x_i-x_j|)], \quad \text{(A10)}$$

$$r_2(x_i,x_j) = \frac{\pi}{12x_i^3x_j^3}[108[\mathrm{Ei}(-2(x_i+x_j)) - \mathrm{Ei}(-2|x_i-x_j|)] + 2\exp(-2|x_i-x_j|)(6(3-x_ix_j)|x_i-x_j| + (x_i^2+3)(x_j^2+3) - 9x_ix_j) - 2\exp(-2(x_i+x_j))(6(3+x_ix_j)(x_i+x_j) + (x_i^2+3)(x_j^2+3)+9x_ix_j) - \exp(-|x_i-x_j|)(12(3-x_ix_j)|x_i-x_j| + 4(x_i^2+3)(x_j^2+3) - 36x_ix_j - 3\ln 3([x_i^2x_j^2+3(x_i-x_j)^2+18]|x_i-x_j| + 4x_i^2x_j^2 - 3x_ix_j(x_i^2+x_j^2)+9(x_i-x_j)^2+18)) + \exp(-(x_i+x_j))(12(3+x_ix_j)(x_i+x_j) + 4(x_i^2+3)(x_j^2+3) + 36x_ix_j - 3\ln 3([x_i^2x_j^2+3(x_i+x_j)^2+18](x_i+x_j) + 4x_i^2x_j^2 + 3x_ix_j(x_i^2+x_j^2) + 9(x_i+x_j)^2+18)) + 3\exp(-|x_i-x_j|)\mathrm{Ei}(-|x_i-x_j|)([x_i^2x_j^2+3(x_i-x_j)^2+18]|x_i-x_j| + 4x_i^2x_j^2 - 3x_ix_j(x_i^2+x_j^2)+9(x_i-x_j)^2+18) - 3\exp(-(x_i+x_j))\mathrm{Ei}(-(x_i+x_j))([x_i^2x_j^2+3(x_i+x_j)^2+18](x_i+x_j) + 4x_i^2x_j^2 + 3x_ix_j(x_i^2+x_j^2)+9(x_i+x_j)^2+18) + 3\exp(|x_i-x_j|)\mathrm{Ei}(-3|x_i-x_j|)(-[x_i^2x_j^2+3(x_i-x_j)^2+18]|x_i-x_j| + 4x_i^2x_j^2$$

$$- 3x_ix_j(x_i^2 + x_j^2) + 9(x_i - x_j)^2 + 18) - 3\exp(x_i + x_j)\mathrm{Ei}(- 3(x_i + x_j))(- [x_i^2x_j^2 + 3(x_i + x_j)^2 + 18](x_i + x_j) + 4x_i^2x_j^2 + 3x_ix_j(x_i^2 + x_j^2) + 9(x_i + x_j)^2 + 18)], \tag{A11}$$

where $\mathrm{Ei}(-x)$ is the exponential integral function (19).

Using these expressions together with the orthogonality of spherical harmonics

$$\int_0^{\pi}\int_0^{2\pi} d\theta\, d\omega \sin\theta\, Y^*_{l_1m_1}(\theta,\omega)Y_{l_2m_2}(\theta,\omega) = \delta_{l_1l_2}\delta_{m_1m_2}, \tag{A12}$$

the second term truncation of Baranger-Mozer contribution (14) takes the form

$$\frac{n^2}{2!}h_2(u) = -\frac{15}{(8\pi)^{3/2}y^3}\sum_{l=0}^{\infty}(-1)^l(2l+1)\int_0^{\infty}\int_0^{\infty} dx_1\, dx_2\, x_1^2x_2^2\left[J_l\left(\frac{y^2u}{x_1^2}\right) - \delta_{l0}\right]\left[J_l\left(\frac{y^2u}{x_2^2}\right) - \delta_{l0}\right]\left[f_l(x_1,x_2) - \frac{y^3}{15}(8\pi)^{1/2}r_l(x_1,x_2)\right]. \tag{A13}$$

2. Third term truncation of the Baranger and Mozer series

Substituting Eq. (18) into Eq. (15) and using expansions Eqs. (A1) and (A2) one finds that the first three terms in Eq. (18) give rise to the following contribution:

$$\frac{n^3}{3!}h_3'(u) = \frac{15}{32\sqrt{2}\pi^2y^3}\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty}\xi_{l_1l_2l_3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} dx_1\, dx_2\, dx_3\, x_1^2x_2^2x_3^2\left[J_{l_1}\left(\frac{y^2u}{x_1^2}\right) - \delta_{l_10}\right]\left[J_{l_2}\left(\frac{y^2u}{x_2^2}\right) - \delta_{l_20}\right] \times\left[J_{l_3}\left(\frac{y^2u}{x_3^2}\right) - \delta_{l_30}\right]f_{l_2}(x_1,x_2)f_{l_3}(x_1,x_3), \tag{A14}$$

where it is introduced

$$\xi_{l_1l_2l_3} = i^{l_1+l_2+l_3}(2l_1+1)^{1/2}(2l_2+1)^{1/2}(2l_3+1)^{1/2}\int_0^{\pi}\int_0^{2\pi} d\theta\, d\omega \sin\theta\, Y_{l_10}(\theta,\omega)Y_{l_20}(\theta,\omega)Y_{l_30}(\theta,\omega). \tag{A15}$$

The integration over the product of three spherical harmonics is very well known in quantum mechanics and expressed through the so-called $3j$-symbol of Wigner as follows:

$$\xi_{l_1l_2l_3} = (-1)^p\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{\sqrt{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2, \tag{A16}$$

where the $3j$-symbol of Wigner in this particular case takes the following form:

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} (-1)^p\left[\dfrac{(l_1+l_2-l_3)!(l_1-l_2+l_3)!(-l_1+l_2+l_3)!}{(2p+1)!}\right]^{1/2}\times\dfrac{p!}{(p-l_1)!(p-l_2)!(p-l_3)!}, \\ \text{if } |l_1-l_2| \leqslant l_3 \leqslant l_1+l_2 \text{ and if } 2p = l_1+l_2+l_3 \text{ is even}, \\ 0, \text{ in all other cases}. \end{cases} \tag{A17}$$

Similar procedure for the convolution term on the right-hand side of Eq. (18) for the triplet correlation function produces the following outcome:

$$\frac{n^3}{3!}h_3''(u) = -\frac{5}{128\sqrt{2}\pi^3y^3}\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty}\xi_{l_1l_2l_3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} dx_1\, dx_2\, dx_3\, x_1^2x_2^2x_3^2\left[J_{l_1}\left(\frac{y^2u}{x_1^2}\right) - \delta_{l_10}\right]\left[J_{l_2}\left(\frac{y^2u}{x_2^2}\right) - \delta_{l_20}\right]\left[J_{l_3}\left(\frac{y^2u}{x_3^2}\right) - \delta_{l_30}\right] \times\int_0^{\infty} dx_4\, x_4^2 f_{l_1}(x_1,x_4)f_{l_2}(x_2,x_4)f_{l_3}(x_3,x_4). \tag{A18}$$

The final expression for the three perturber contribution is ultimately found as a sum of Eq. (A14) and Eq. (A18),

$$\frac{n^3}{3!}h_3(u) = \frac{n^3}{3!}h_3'(u) + \frac{n^3}{3!}h_3''(u). \tag{A19}$$

One should notice here that Eqs. (A14) and (A18) contain three and four dimensional integrals that are hard to evaluate numerically but it is symmetry in the coefficients $\xi_{l_1l_2l_3}$ that allows one to reduce those integrations to two dimensions.

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